

Plane Symmetric Solutions in $f(R)$ Gravity

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Abstract

The modified theories of gravity, especially the $f(R)$ theory, have attracted much attention in recent years. In this context, we explore static plane symmetric vacuum solutions using the metric approach of this theory. The field equations are solved using the assumption of constant scalar curvature which may be zero or non-zero. We have found a total of three plane symmetric solutions. The correspondence of these solutions with the well-known solutions in General Relativity is given.

Keywords: $f(R)$ gravity; Plane symmetric solutions.

1 Introduction

It is a reality that general theory of relativity (GR) has resolved many problems since its birth. However, there are some issues which are still problematic. For example, the localization of energy is still an irritating issue. Also, there are some other problems in astrophysics and cosmology like the issues of dark energy and dark matter where we experience severe theoretical

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difficulties. It has been found that 96 percent energy of the universe contains dark energy and dark matter (76 percent dark energy and 20 percent dark matter) [1]. Dark matter is basically an unknown form of matter which has the same properties as ordinary matter but cannot be detected in the laboratory.

Singularity is another important issue in GR. The singularity theorem shows that occurrence of the spacetime singularity is a general feature of any cosmological model under some reasonable conditions. It may be possible to avoid these undesired singularities in the context of alternative theories. For example, it has been shown [2] that there does not exist any cosmological singularity by considering higher order curvature terms. Thus there is a need of modified or generalized theories to tackle these sort of problems. The $f(R)$ theory of gravity is one of the modified theories which may help to resolve such issues.

The $f(R)$ actions were first studied by Weyl [3] and Eddington [4] in 1919 and 1922 respectively. Later, Buchdahl [5] studied these actions rigorously in the context of non-singular oscillating cosmologies. Jakubiec and Kijowski [6] worked on theories of gravitation with non-linear Lagrangian. They proved that any theory of gravitation with a non-linear Lagrangian depending on the Ricci tensor was equivalent to the Einstein theory of gravitation interacting with additional matter fields. Recent literature [7]-[11] shows keen interest in exploring different issues in $f(R)$ theories of gravity. Sotiriou [7] investigated the relationship between $f(R)$ theory of gravity and scalar tensor theory. Amendola et al. [8] derived the conditions under which dark energy $f(R)$ models are cosmologically viable. Hu and Sawicki [9] verified that the choices of $f(R)$ models have stable high-curvature limits and well-behaved cosmological solutions with a proper era of matter domination only if $d^2 f/dR^2 > 0$.

The most widely explored exact solutions in $f(R)$ gravity are the spherically symmetric solutions. Multamäki and Vilja [10] studied spherically symmetric vacuum solutions in this theory. They found that the whole set of the field equations in $f(R)$ gravity gave exactly the Schwarzschild de Sitter metric. The same authors [11] investigated the perfect fluid solutions and showed that pressure and density did not uniquely determine $f(R)$. It was found that matching the outside Schwarzschild de Sitter metric to the metric inside the mass distribution led to additional constraints that severely limited the allowed fluid configurations. Capozziello et al. [12] explored spherically symmetric solutions of $f(R)$ theories of gravity via the Noether symmetry

approach. Hollenstein and Lobo [13] analyzed the exact solutions of static spherically symmetric spacetimes in $f(R)$ modified theories of gravity coupled to non-linear electrodynamics.

It is interesting, at least from theoretical point of view, to consider other exact solutions of the field equations in $f(R)$ gravity. Recently, Azadi et al. [14] have studied cylindrically symmetric vacuum solutions in metric $f(R)$ theories of gravity. Momeni and Gholizade [15] extended this work to a more general cylindrically symmetric solution. It would be interesting to extend this analysis to plane symmetric vacuum solutions.

In this paper, we focuss our attention to investigate the exact solutions of static plane symmetric (for a class of plane symmetric spacetimes for which the coefficient of $dx^2 = 1$) vacuum solutions in $f(R)$ theories of gravity using metric approach. In particular, we have found three solutions by using the assumption of constant scalar curvature. The paper is organized as follows: In section 2, we give a brief introduction about the field equations in the context of $f(R)$ gravity. Section 3 is used to find plane symmetric solutions and also the constant curvature solutions. In the last section, we summarize and conclude the results.

2 Some Basics of $f(R)$ Gravity

There are mainly two approaches in $f(R)$ theory of gravity. The first is called "metric approach" in which the connection is the Levi-Civita connection and the variation of the action is done with respect to the metric. The second approach is the "Palatini formalism" in which the connection and the metric are considered independent of each other and the variation is done for the two parameters independently. Here we deal the problem with the metric approach.

The $f(R)$ theory of gravity is basically the modification or generalization of general theory of relativity. The action for $f(R)$ gravity is given by [10]

$$S = \int \sqrt{-g} \left(\frac{1}{16\pi G} f(R) + L_m \right) d^4x. \quad (1)$$

Here $f(R)$ is a general function of the Ricci scalar and L_m is the matter Lagrangian. It may be observed that this action is obtained by just replacing R with $f(R)$ in the standard Einstein-Hilbert action. The corresponding field

equations are found by varying the action with respect to the metric $g_{\mu\nu}$

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \square F(R) = \kappa T_{\mu\nu}, \quad (2)$$

where

$$F(R) \equiv df(R)/dR, \quad \square \equiv \nabla^\mu \nabla_\mu \quad (3)$$

with ∇_μ the covariant derivative and $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the Lagrangian L_m . These are the fourth order partial differential equations in the metric tensor. The fourth order is due to the last two terms on the left hand side of the equation. If we take $f(R) = R$, these equations reduce to the field equations of GR.

Now contracting the field equations, it follows that

$$F(R)R - 2f(R) + 3\square F(R) = \kappa T. \quad (4)$$

In vacuum, this reduces to

$$F(R)R - 2f(R) + 3\square F(R) = 0. \quad (5)$$

This gives a relationship between $f(R)$ and $F(R)$ which may be used to simplify the field equations and to evaluate $f(R)$. It is obvious from this equation that any metric with constant scalar curvature, say $R = R_0$, is a solution of the contracted equation (5) as long as the following equation holds

$$F(R_0)R_0 - 2f(R_0) = 0. \quad (6)$$

This is called constant curvature condition. Moreover, if we differentiate Eq.(5) with respect to R , we obtain

$$F'(R)R - R'F(R) + 3(\square F(R))' = 0. \quad (7)$$

which gives a consistency relation for $F(R)$.

3 Plane Symmetric Vacuum Solutions

Here we shall find plane symmetric static solutions of the field equations in $f(R)$ gravity. For the sake of simplicity, we take the vacuum field equations and also use constant scalar curvature ($R = \text{constant}$). In the following, we obtain three particular solutions of the static plane symmetric spacetimes with these conditions in $f(R)$ gravity.

3.1 Plane Symmetric Spacetimes

We consider the general static plane symmetric spacetime given by

$$ds^2 = A(x)dt^2 - C(x)dx^2 - B(x)(dy^2 + dz^2). \quad (8)$$

For the sake of simplicity, we take $C(x) = 1$ so that the above spacetime takes the form

$$ds^2 = A(x)dt^2 - dx^2 - B(x)(dy^2 + dz^2). \quad (9)$$

The corresponding Ricci scalar becomes

$$R = \frac{1}{2} \left[\frac{2A''}{A} - \left(\frac{A'}{A} \right)^2 + \frac{2A'B'}{AB} + \frac{4B''}{B} - \left(\frac{B'}{B} \right)^2 \right], \quad (10)$$

where prime represents derivative with respect to x . Using Eq.(5), it follows that

$$f(R) = \frac{3\Box F(R) + F(R)R}{2}. \quad (11)$$

Inserting this value of $f(R)$ in the vacuum field equations, we obtain

$$\frac{F(R)R_{\mu\nu} - \nabla_\mu \nabla_\nu F(R)}{g_{\mu\nu}} = \frac{F(R)R - \Box F(R)}{4}. \quad (12)$$

Since the metric (9) depends only on x , one can view Eq.(12) as the set of differential equations for $F(x)$, A and B . It follows from Eq.(12) that the combination

$$A_\mu \equiv \frac{F(R)R_{\mu\mu} - \nabla_\mu \nabla_\mu F(R)}{g_{\mu\mu}}, \quad (13)$$

is independent of the index μ and hence $A_\mu - A_\nu = 0$ for all μ and ν . Thus $A_0 - A_1 = 0$ gives

$$\left[\frac{A'B'}{AB} + \left(\frac{B'}{B} \right)^2 - \frac{2B''}{B} \right] F - \frac{1}{2A} (A'F' + 2F'') = 0. \quad (14)$$

Also, $A_0 - A_2 = 0$ yields

$$\left[\frac{2A''}{A} - \left(\frac{A'}{A} \right)^2 + \frac{A'B'}{AB} - \frac{2B''}{B} \right] F - \frac{1}{2} \left(\frac{A'}{A} - \frac{B'}{B} \right) F' = 0. \quad (15)$$

Thus we get two non-linear differential equations with three unknowns namely A , B and F . The solution of these equations could not be found straightforwardly. However, we can find a solution using the assumption of constant curvature.

3.2 Constant Curvature Solutions

For constant curvature solution, say for $R = R_0$, we have

$$F'(R_0) = 0 = F''(R_0). \quad (16)$$

Using this condition, Eqs.(14) and (15) reduce to

$$\frac{A'B'}{AB} + \left(\frac{B'}{B}\right)^2 - \frac{2B''}{B} = 0, \quad (17)$$

$$\frac{2A''}{A} - \left(\frac{A'}{A}\right)^2 + \frac{A'B'}{AB} - \frac{2B''}{B} = 0. \quad (18)$$

These equations are solved by the power law assumption, i.e., $A \propto x^m$ and $B \propto x^n$, where m and n are any real numbers. Thus we use $A = k_1 x^m$ and $B = k_2 x^n$, where k_1 and k_2 are constants of proportionality. It follows that

$$m = -\frac{2}{3}, \quad n = \frac{4}{3} \quad (19)$$

and hence the solution becomes

$$ds^2 = k_1 x^{-\frac{2}{3}} dt^2 - dx^2 - k_2 x^{\frac{4}{3}} (dy^2 + dz^2). \quad (20)$$

It can be shown that these values of m and n lead to $R = 0$. We re-define the parameters, i.e., $\sqrt{k_1} t \longrightarrow \tilde{t}$, $\sqrt{k_2} y \longrightarrow \tilde{y}$ and $\sqrt{k_2} z \longrightarrow \tilde{z}$, the above metric takes the form

$$ds^2 = x^{-\frac{2}{3}} d\tilde{t}^2 - dx^2 - x^{\frac{4}{3}} (d\tilde{y}^2 + d\tilde{z}^2) \quad (21)$$

which is exactly the same as the well-known Taub's metric [16].

Now we assume $A(x) = e^{2\mu(x)}$ and $B(x) = e^{2\lambda(x)}$ so that the spacetime (9) takes the form

$$ds^2 = e^{2\mu(x)} dt^2 - dx^2 - e^{2\lambda(x)} (dy^2 + dz^2). \quad (22)$$

The corresponding Ricci scalar is given by

$$R = 2\mu'' + 2\mu'^2 + 4\mu'\lambda' + 4\lambda'' + 6\lambda'^2. \quad (23)$$

Using Eq.(13), $A_0 - A_1 = 0$ and $A_0 - A_2 = 0$ respectively yield

$$2(\lambda'' + \lambda'^2 - \mu'\lambda')F - \mu'F' + F'' = 0, \quad (24)$$

$$(\mu'' - \lambda'' + \mu'^2 - 2\lambda'^2 + \mu'\lambda')F + (\mu' - \lambda')F' = 0. \quad (25)$$

For constant curvature solutions, the above equations reduce to

$$\lambda'' + \lambda'^2 - \mu'\lambda' = 0, \quad (26)$$

$$\mu'' - \lambda'' + \mu'^2 - 2\lambda'^2 + \mu'\lambda' = 0. \quad (27)$$

Equation (26) can be written as

$$\lambda' \left(\frac{\lambda''}{\lambda'} + \lambda' - \mu' \right) = 0 \quad (28)$$

which leads to the following two cases:

$$I. \quad \lambda' = 0, \quad II. \quad \frac{\lambda''}{\lambda'} + \lambda' - \mu' = 0.$$

Now we solve the field equations for these two cases.

Case I

It follows from the case I that

$$\lambda = a, \quad (29)$$

where a is an integration constant. Inserting this value in Eq.(27) and integrating the resulting equation, we obtain

$$\mu = \ln(bx + bc), \quad (30)$$

where b and c are integration constants. Thus the metric (22) becomes

$$ds^2 = (bx + bc)^2 dt^2 - dx^2 - e^{2a}(dy^2 + dz^2). \quad (31)$$

The corresponding scalar curvature is

$$R = 0. \quad (32)$$

It is mentioned here that the metric (31) is a solution only for those $f(R)$ functions which are linear superposition of R^m . For instance, $f(R) = R + aR^2$ could be the right choice and $f(R) = R - (-1)^{n-1} \frac{a}{R^n} + (-1)^{m-1} bR^m$ cannot be used in this case. The reason is that the function of Ricci scalar becomes undefined for $R = 0$. This solution corresponds to the self-similar solution of the infinite kind for the parallel dust case [17].

Case II

The second case yields

$$\mu' = \lambda' + \frac{\lambda''}{\lambda'}. \quad (33)$$

Integrating this equation, we obtain

$$\mu = \lambda + \ln \lambda' + d. \quad (34)$$

Using the assumption of constant scalar curvature, it follows that

$$2\frac{\lambda'''}{\lambda'} + 14\lambda'' + 12\lambda'^2 = \text{constant} \quad (35)$$

which is a third order non-linear differential equation. The general solution of this equation seems to be difficult. However, a special choice, out of a larger set of possible solutions is that $\lambda(x)$ is a linear function of x , i.e., $\lambda(x) = fx + g$, where f and g are arbitrary constants. Consequently, the metric (22) takes the form

$$ds^2 = e^{2(fx+\bar{g})}dt^2 - dx^2 - e^{2(fx+g)}(dy^2 + dz^2), \quad (36)$$

where $\bar{g} = g + \ln f + d$. The corresponding Ricci scalar reduces to

$$R = 12f^2. \quad (37)$$

Now re-defining $e^{\bar{g}}t \longrightarrow \tilde{t}$, $e^g y \longrightarrow \tilde{y}$ and $e^g z \longrightarrow \tilde{z}$, it follows that

$$ds^2 = e^{2fx}(d\tilde{t}^2 - d\tilde{y}^2 - d\tilde{z}^2) - dx^2. \quad (38)$$

This corresponds to the well-known anti-de Sitter spacetime [18] in GR. Thus we have found a total of three static plane symmetric solutions with the assumption of constant scalar curvature which may be zero or non-zero in $f(R)$ gravity.

4 Summary and Conclusion

The purpose of this paper is to study the plane symmetric solutions in $f(R)$ theories of gravity. We have used the metric approach of this theory to study the field equations. Since the field equations in this theory are highly non-linear and complicated, it is very difficult to solve them analytically without

any assumption. We have developed an important condition, given by Eq.(6), for constant scalar curvature.

We can assume the function F arbitrarily to solve these equations but this gives fourth order highly non-linear differential equations. The assumption of constant curvature (may be zero or non-zero) is found to be the most suitable and we can get some solution for constant scalar curvature. However, it is not always guaranteed that solution would be possible using this assumption. Using this assumption, we have found a total of three static plane symmetric solutions. It is mentioned here that out of these three solutions, two are exactly similar to Taub's solution and anti deSitter spacetime while the third solution corresponds to the self-similar solution already available in the literature. The physical relevance of these solutions is very much obvious.

It has been shown [19] that dark energy and dust matter phases can be achieved by the exact solution derived from $f(R)$ cosmological model. The dark energy is considered as one of the causes for expansion of universe. In a recent paper [20], we have discussed the expansion of universe in the context of metric $f(R)$ gravity. Thus we expect that such solutions may provide a gateway towards the solution of dark energy and dark matter problems. We would like to mention here that most of the work in $f(R)$ gravity has been done for vacuum static cases. It would be worthwhile to investigate solutions for non-static and non-vacuum cases.

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